What is Mathematics?

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It is good to be with you all today and to have a chance to practice philosophy with you.

For the next few minutes I am going to jump in to the philosophy of mathematics—what some people call the *foundations of mathematics—asking the big question 'what is mathematics?'—working through some of the classical and newer arguments in the field.*

At the end of my talk today I am hoping that I will have laid out some good questions for discussion—also to get some critical feedback regarding my main thesis on the question. Between now and then let me jump into the arguments and see where the evidence leads.

Let's begin with some history on the question. Some traditional answers to the question 'what is mathematics?' are that mathematics is about abstractions; about mental con- structions; about symbols; or hypothetico-deductive systems; or that it reduces to logic. None of these ideas ever entirely won the day, or overcame objections, or satisfied critics—the jury is still out, as it has been for the past several millennia, and has pretty much stayed where it is now since Gödel first published his Incompleteness theorem, in 1931.

So let's catch up with the discussion and bring it up to date to Gödel, and what has happened since his time, including some very recent attempts to 'unify' mathematics.

Pythagoras must have thought that the world was made of mathematics in some sense—made of numbers—numbers conceived as assemblies and ratios of units of magnitude— which is why this whole way of thinking fell apart immediately after Pythagoras' later discovery of irrational numbers, which cannot be expressed as whole number ratios.

Plato was drawn more to the process of abstraction behind the conception of number—not five oranges or pencils or people but the abstract notion 'five' they have in common. Plato originates the tradition of answering the question 'what is mathematics?' by talking about a special kind of object called a "form" (*eidos*) or abstract object. Thus among the things that exist are mathematical objects; these are abstract (non- spatiotemporal); and they exist independently of intelligent agents. This view is variously called Platonism or Mathematical Platonism or Object Realism. The basic 'realist' claim is that the mind *discovers* inhering structures in the universe—'forms' or 'ideas'—rather than *inventing* them.
Thus we invent symbols— which are arbitrary—and by using them we discover realities— which imply necessity—we invent the language but discover the structure of reality. Some of the things we discover to be real are material and concrete. Some are immaterial and abstract.
Thus mathematics is about the real world but more narrowly is about *the most abstract* elements of reality.
Mathematics is the science of pure thought.

Plato introduces the term *chorismos* (separation) to help identify what he is talking about. He says that the way we learn mathematical principles shows us their purely noetic quality— they are separate—they cannot be sensed but are intelligible—they are seen and comprehended by being demonstrated—and so because they are the main examples of what we can learn about the world when we try to *demonstrate* exactly what we know, they are not just *ta pragmata*, things insofar as we have to do with them practically, but *ta mathemata*, things insofar as we are able to learn about them—mathematical things—things that we are able to demonstrate and to teach others exactly as *we* have learned.

Aristotle's view is that abstract properties reside in the objects from which they are abstracted. This means that mathematics is not about something *chorismos* or separate. Aristotle is not disputing the import or truth of results in mathematics, but the nature of mathematical thinking, “so that our controversy is not about their being but about their mode” (*Metaphysics* 1076 a 36). He argues that mathematical truths are not separate from but are *dependent on* human beings. He argues that a number, e.g. three, has no independent nature, but is just a way of talking about something being “so many,” as for example so many trees (*Metaphysics* 1080 a 15). In the *Physics* he expresses this idea by saying that “to be a number is to be some number of given things” (221 b 14). He argues that the *chorismos* thesis implies that—since they are separate—abstract objects would have to exist prior to sensible things, but he thinks it makes much more sense to think that sensible things exist before anything can be counted and thus have a “number” (*Metaphysics* 1077 a 15).

Aristotle thinks Plato is making the same mistake in jumping from five oranges and five people to the abstract idea ‘five’ as he makes when he jumps from ‘man’ to ‘men’ to ‘Man.’ There is no idea of Man separate from human beings—this is just a confusing way of talking about properties of the biological species ‘man.’ This is just a way a talking and does not imply any metaphysics—it’s just semantics.

When we talk about ‘mathematics’ this is just shorthand for properties in the real world. Thus, beginning from Plato’s ‘realism,’ Aristotle offers the first ‘nominalist’ correction.

This is the background for Aristotle’s discussions about first principles in the *Organon* and in the first chapters of the *Physics,* where he discusses what we should consider real and that reality cannot be *one* in the sense in which Parmenides argued (185a 20).

According to tradition, Euclid’s discussion of first principles in the first book of the *Elements* continues Aristotle’s line of thinking in the *Organon*. What Aristotle calls "hypotheses of existence" Euclid divides into definitions (*horoi*), postulates (*aitêmata*), and common notions (*koinai ennoiai*). In a sense, Euclid is collecting together in one place everything that people had learned about mathematics in all the ancient schools—the Pythagoreans, the Eleatics, Platonism from the Academy and Aristotle's Lyceum. Euclid develops *the* classic example of developing a structure deduced from principles, so much so that for medieval thinkers like John of Salisbury, Walter Burleigh and Occam, mathematics simply is the Euclidean demonstrative method from basic axioms.

What impressed Euclid's more distant successors was not his demonstrative system but his building process itself—thinking of mathematics very narrowly as the process Euclid undertakes \e.g. in his first proposition: Book I Proposition 1—to construct an equilateral triangle from a given line segment. Euclid does not say whether or not a triangle exists; he begins talking about points and lines as a matter of course and applies his postulates until the thing is done (*hoper edei poiesai*, L. *quod fieri*) or the statement is proved (*hoper edei deixai, L. quod erat demonstratum*). Thinkers like Brouwer and Heyting developed so-called "intuitionist" mathematics from Euclid's process of gradually assembling an object of study (in his words "to poetize"—*poiesai,* 'make,' and *deixai,* 'show,' 'bring to light') in order to bring it about and make use of it—thus e.g. to show geometrically how to build an equilateral triangle—a view called *constructivism* today.

After the classical age, the great names in mathematics emerge in the Islamic world—Al-Khwarizmi, Alhazen, Omar Khayyam—from whom we get zero, negative numbers, letters to symbolize quantities of different kinds, the use of the unknown, the method of rebalancing equations and cancelling terms, and the general strategy of reduction to simplest terms, which came to be known by Al Khwarizmi's name—algorithmic compression—thus laying out the first principles of algebra (Ar. *al-jabr*, 'reunion of what is broken'). Russell once explained in a beautiful phrase that "in algebra the mind is first taught to consider general truths" and that the whole point of mathematical education is to hone exactly this ability to deal with strictly abstract truths. Alhazen makes the jump to experimental reasoning and Fibonacci introduces the Hindu-Arabic number system to the West—together laying the foundations of the scientific method.

This brief history—antedating the modern world— motivates mathematical realism, nominalism, logicism and constructivism—the foundational ideas in the subject. This demonstrates for us that the philosophy of mathematics is dominated by ideas from its history, before the revolutionary work in mathematics begins in the Enlightenment.

The new language of algebra made it possible to collect all mathematical knowledge within a single language—what Descartes calls 'analytic geometry'—a language made entirely of abstract expressions. But then this form itself opens up as an object of study—i.e., thinking began to abstract from abstractions. Thus in a sense Euclid abstracts from experience in conceiving the idea of a mathematical point, but Bombelli (1572) is simply experimenting with ways of finding roots for equations when he stumbles on the idea of the imaginary number. He is trying to think about the number line extended into 2D and conceives of vectors in the complex plane. Here I am trying to make the point that the history of mathematics is driven by formal problems—in response, thinking creates new tools—these new tools then inauguate a new round of thinking about foundations.

Leibniz fills many pages of notes with approaches for calculating the area under a curve— in 1675 he develops a fresh approach with some definitions—he defines the idea of a constant; of a variable; of a dependent variable and an independent variable; and the idea of the *function* that relates them. He defined the *derivative* of a function as the rate of change of a function with regard to its independent variable. Thus a function y of a variable x, like y = f (x) takes various values given various inputs, which he conceives as ordered pairs (x,y) of points on a Cartesian plane—later as triplets (x,y,z) on a 3D grid. He defines the *derivative* of a function f ?(x) = dy / dx—the change of y over the change of x—and defines the *differential* as any such arbitrary change in the value of x, thus computing the differential dy = f ? (x) dx. Extrapolating, he conceives the differential as an *infinitesimally small* change in value, which he expresses mathematically as the symbol d as we do today. He conceives the integral, symbolized with the elongated s symbol ? as we do today, as the sum of a sequence of subdivisions, i.e. of differentials, as the anti-derivative, an idea known today as the Fundamental Theorem of the Calculus. Thus the area under a curve is the integral of the differential slices under it to the x-axis:

A=?*f(x)dx*

Newton, who also discovered these “principles of calculus” independently, immediately saw that they have physical meaning. If we think of a point moving in space as a variable y coordinate along an extending x-axis, the rate of change of the area bounded by the curve is this very curve itself. Thus the tangent to the curve at any arbitrary point is its instantaneous rate of change—an infinitesimal increment of time—the derivative f ? (x) is the slope of the tangent to the curve y = f (x) at any arbitrary point on the abscissa.

Newton models the physical universe as made up of bodies each having some form of attraction to one another, as natural weight or mass or heaviness (in Latin *gravitas*), as the earth attracts the moon, the sun the earth, and an apple falls to the center of the earth. By observation, neither the moon nor the apple moves in a straight line at constant speed. Therefore an attractive force exerts an *acceleration* on a distant body (an important form of which is the acceleration of falling objects near the surface of a body due to gravity). Now velocity is the rate of change of distance with respect to time—the derivative of the distance function (rate times time equal distance)— acceleration is the rate of change of velocity with respect to time—the second derivative of the distance function—and so we have constructed mathematical tools for describing what Newton calls “the system of the world” interrelating any object in the world, the forces acting upon it, and its motions.

That is:

? F = 0 ? dv/dt = 0 [law of inertia]

F = m dv/dt = ma[impulse, instantaneous application of force, is (J = ? F dt)] Fa = - Fb [to every action an opposite and equal reaction]

That is:

An object at rest will stay at rest unless a force acts upon it— an object in motion will not change its velocity unless some force acts up it. Force is the derivative of momentum over time—momentum is mass * velocity—thus force is mass * acceleration. These are what Newton calls the “mathematical principles of natural philosophy”—*Philosophiae Naturalis Principia Mathematica*—a work first published in 1686. The *Principia* defines philosophy as “arguing from phenomena to investigate the forces of nature.” Newton says that he is *cultivating mathematics (excolo)* in so far as it relates to philosophy, which implies that mathematics is something we have to work on to practice and to improve.

Newton remarks that his discovery of universal gravitation demonstrates that things that seemed to have nothing to do with one another—falling objects and the motion of the planets—were forms of one principle. Magnetism and electricity were later understood to be forms of one thing— electromagnetic (EM) waves—light was later recognized to be an undulation of EM fields—space and time were seen to be aspects of the same continuum—spacetime—ST— then gravity was seen to be a kind of curvature of ST. Thus from the initial problem of finding roots for algebraic equations—a set of problems inherited from Al Khwarizmi—Leibniz and Newton ended up inventing an entirely new language for understanding the physical universe—the basic mathematics of change.

Even within his lifetime, Newton’s principles were applied to astronomy, mechanics, civil engineering, metallurgy, chemistry, hydraulics, hydrostatics, optics, lens grinding, architecture, ship-building, and the design of cannons, bridges and clocks.

This raises the following fascinating question—why has this mathematical method of regarding nature been so spectacularly successful? This method arose out of nowhere simply through trying to respect the basic rules for maintaining algebraic closure over basic operations.

How can we explain the amazing match between pure theory and practical applications?

The tradition of modern German philosophy, beginning with Leibniz and Kant, and carried on by writers like Husserl, Weber, Jaspers, Heidegger and Hannah Arendt, make this the central question in the philosophy of mathematics—Arendt concludes her studies with the admission that *the* greatest perplexity in the entire state of affairs we are looking at is that completely fictional and arbitrary ideas can and do describe actual physical processes and thus can be seen to "work." She says: it's as if our methods outpace our understanding—mathematical truth may not even be comprehensible to human reason.

And so with Newton we have a second, major puzzle in the philosophy of mathematics—something on the order of Pythagoras' terrifying discovery of irrational numbers.

Kant thinks he knows the answer to this question. He thinks that he has learned it from Galileo. Galileo says famously that the laws of nature are written in the language of mathematics. Kant agrees and says even further that there is only so much science in a thing—that is, *knowledge* of a thing—as there is mathematics in it. But Kant says that he is making a Copernican turn—overturning the basic premises of understanding. Thus he says it is not that the laws of nature are written in the language of mathematics, but that we must bring a new template for understanding reality by thinking mathematically. "Reason has insight only into that which it produces after its own design" (KdRV B xiii).

Kant's new idea is that mathematical statements are *a priori* synthetic judgments.

That is: mathematics is *a priori*, rather than *a posteriori*—it is 'before' (it is prior, i.e. it must be assumed beforehand) rather than 'after' (meaning what occurs in experience). This 'before' structure becomes the main subject of thinking about mathematics for centuries afterwards—trying to understand mathematics as a mental projection.

Husserl uses many different terms to talk about 'before' structure—"forerunner ideas"—the "already-made world"—the "bases from which" we develop every concept we have. He looks for a way to talk about the primordial pre-reflective laying out of the world, which he conceives as a kind of embedded power coeval with the advent of language.

He sees that the act of creating a structure in a symbolism immediately becomes independent of the objects the structure was intended to model—a brand new development begins within the new medium itself—this 'transformation' is the key. Husserl's student Heidegger calls it "the fundamental presupposition for knowledge": "*Ta mathemata*, mathematical things, are things insofar as we take cognizance of them as what we already know them to be *in advance*. Thus we already know in advance what counts as a physical object, we already know the plant-like character of plants, we already have an idea of what counts as an animal and the animal-like characteristics that alert us that what we are looking at is an animal. Thus the kind of learning that we are talking about—*ta mathemata*, things insofar as they can be learned —- is an extremely peculiar kind of seeing and taking in which we are taking something *we already have*."

Thus the mathematical project *anticipates* the essence of things—it defines the advance blueprint for the structure of things in relation to everything possible—it is the basic guide which at the same time is the fundamental measure for laying out the universe.

Heidegger explains further that the mathematical method of getting at a thing—the mode of access appropriate to axiomatically pre-determined basic planning—works via the cycle of hypothesis-experiment-output, designed to posit conditions in advance and then, by applying reason and, gradually, using instrumentation, to pose key questions that observation alone can answer. For this reason he says that the mathematical project *skips over things* in order to look for facts. This is Kant's great achievement—the achievement of the Enlightenment—the spirit of experimental natural philosophy.

If the question is, why this works?, we can begin to see that in many cases it doesn't—we try something and it doesn't work out—it's just a cycle of experiments and results. In a sense, Kant offers the precedent for getting away from thinking about mathematics as a *body of knowledge*—he wants us to consider the proposition that mathematics is something that human beings do. There is a verb buried somewhere in mathematical thinking—a function or an action—this is what Heidegger means by "skipping over."

But then even this formulation is misleading since it invites us to make an important mistake, that of essentializing into a *single* basic ability or faculty what is in reality an enormously complicated array of techniques that are very different from one another.

The word 'mathematics' is plural in most languages, which suggests that whatever else mathematics may be, it is not one thing, but many. So a next step in reasoning is to accept that mathematics is a big collection—more like a toolbox than a language—which must include things like counting, measuring, abstracting, demonstrating, applying techniques like differentiation or integration or graphing, or taking an average. Going with the toolkit metaphor, let's examine this more closely. The reason we are reaching into the bag is to do something—to solve a problem—so we are looking around for a tool—we have a bunch of tools but we need the right one for the job.

So to begin with there is the world, and we are going to use mathematics to solve a problem in the world, which means that mathematics is something we bring with us to experience—we carry it with us and make use of it when we need to—which is why Plato thinks of mathematics in terms of *anamnesis,* or recollection—calling a thought to mind—just as Aristotle thinks of mathematics in terms of the most basic categories by which we orient ourselves in life—mathematical ideas are forerunner ideas—or overall a set of tools by which we take the measure of a thing, examine it and ask questions.

In a sense what we are talking about here is *modeling*—we are trying to come up with a mathematical model for something that is happening in the world—and so the point of talking about the toolkit is to emphasize that there is always more *than one way to do this.* Look closely at the idea of a mathematical model. A mathematical model is a different thing than the thing it is meant to model. Consider an example. Think of an equilateral triangle T standing upright with vertex a at the top, with vertices denoted . Now rotate the triangle counter-clockwise from the top around its midpoint. It has three upright positions with a vertex pointing upwards: the beginning point 0, then rotated 2 ð over 3, and then rotated again 4 ð over 3 radians. Now if we consider this simple scenario as a *situation in the world*, we could say that in this situation S, in the observable universe, there are three physical states possible—modeled as various abstract states Q. But immediately we see that we can model these three states in the following different ways:

Set Q1 = (, ,) Set Q2
= (0c, 2 ð / 3c, 4 ð / 3c) Set Q3 = (1, 2, 3)

The point is that the identical situation in the universe can be managed by different modeling assumptions. Note that the abstract states could be a number, but might not be a number. From the standpoint of nature, there is no preferred description between Q1, 2, 3 ... but in some cases one kind of Q may make more sense or be more helpful to use. Sometimes it is useful to count; sometimes to imagine. It’s as if: *adopt this convention*, and then see where you get. Note also that the universe goes on all by itself whether we create abstract states to model its observable events or not—the labeling conventions have to do with us, not with what’s out there in the universe. Just as all three Q sets can work to describe a triangle’s rotation around its axis, so we can model extremely complex processes equally well and usefully with dissimilar tools.

I think this is what Cantor meant when he said that the essence of mathematics is freedom. In mathematics we are able to imagine any situation we like and then try to get at it by making wildly different kinds of assumptions. This is why mathematics is so freeing—we don't even have to be constrained by reality— we simply construct a tool and see what it does. Sometimes we look in the kit and reach for the wrong tool—whatever we try doesn't work—yet this is not a problem in itself but is good information—this is the important thesis that Imre Lakatos put forward in the 70s—that it's unhelpful to expect too much or to have too highfalutin' an idea about what we are doing— mathematics is simply heuristics—what we've been taught is wrong—mathematics is *not* about eternal truths—we are simply trying out ideas and seeing what works.

Now in order to see more of what is going on in mathematics and to see more into the nature of mathematical modeling, and maybe also to get a little closer to answering the question, *what is mathematics?*, I thought that it made sense to choose something to look at in the world and then try to build up a mathematical system as a wireframe for it to begin examining it. And so in an earlier draft of this paper, I developed a mathematical model of love—roughly a problem in dynamics, or in systems that evolve in time—as a system of differential equations—which I will not take the time to spell out here today (see notes to this essay following page 10). With a few assumptions, I made a start in modeling the environment in which people fall in love, a frictionless or a resisting medium, and I spelled out some problems with mate choice, romantic styles, the complexity of emotional reactions and influences on mood, general sociability and romantic outcomes, and the likelihood of divorce vs. long-term relationships. Arguably my results have some predictive power and might even be applied in real-time scenarios on the model of a control system. Knowledge like this could perhaps help a person estimate risks and make better decisions in his love life. But I think it would be odd to say that, because of such calculations, I now understand *love* any better. Perhaps I have gotten a little closer to seeing some patterns of love and some of how it works.

I think what is most obvious here is that there is nothing in the mathematics of love that directs *how this knowledge should be used*—evidently mathematical tools can be used for good or ill and, as nearly all the philosophers seem to tell us, from Plato through Grothendieck, the deep problem in mathematics is the moral one—i.e. what this knowledge is for.

At this point I think I can offer a helpful analogy. In physics there is something called "the Copenhagen interpretation," which is something that Neils Bohr and Werner Heisenberg created in the 1920s to help people interpret all the new data being gathered about quantum processes and how to go about controlling them. This interpretation was accepted at the time and is widely accepted today as *the* working model for understanding interactions at the smallest scales we can measure. Yet the Copenhagen interpretation remains fundamentally problematic in that it cannot explain or reconcile the particle- wave duality—it basically just gives up on this question. So we have a useful scientific theory that allows us to make predictions that are confirmed by experiments, but one that is philosophically unsatisfying—more accurately, it is philosophically *incoherent*. We have a method that works but no real understanding of what it means. I think this is roughly the situation we are in with mathematics today—mathematics is various methods and a long list of different kinds of heuristics, theses, and formalisms—thus we have methods that work but no satisfying comprehensive idea of what mathematics is.

Probably the most important single result in contemporary mathematics is Gödel's discovery of the incompleteness principle. Gödel noted (as we have above) that the entire history of mathematics takes the position that mathematics is something *a priori*—it represents the basic instruction-set or blueprint for the universe and therefore is something that we ourselves bring to experience, as a basic condition, before we actually 'have' any experience at all. Thus as Wittgenstein said there can be no surprises in mathematics. But Gödel's discoveries did come as a surprise. It is surprising to learn that a mathematical proposition might be true even though there is *no possible way* of proving it. As it turns out, we know things we cannot prove—no system of axioms is capable of generating all the truths of mathematics—which Gödel interpreted to mean that we have a kind of *intuition* for truths that sweeps beyond the reach of any logical system. Ultimately mathematics is about this power rather than any of its creations.

Gödel’s thesis precipitated the third crisis in the foundations of mathematics—the crisis of incompleteness—inspiring many ‘rescue’ attempts to unify mathematics on new principles—just as Pythagoras’ problems with irrationals and Leibniz’s problems with infinity drove the earlier history of mathematics—from which we get e.g. Bombelli’s imaginary numbers and Cantor’s set theory—some recent attempts are Category theory from the 1940s, Grothendieck’s Scheme theory from the 1960s and the Langlands program from the 1970s.

For the present, there is no consensus about the subject—we are pretty much where Gödel left us. But I think Gödel has given us some ideas about how to think about mathematics as the big collection—the toolkit—which I have been calling the Grab Bag Mind—the GBM—for a while. The GBM is full of funny little tricks and schemes, which to me seem like data-collection devices of different sorts.

Some things about the GBM that seem important to me—and I guess this is my thesis—are that it’s just a bunch of resources. It’s not a language—Galileo was wrong about this. It’s not a portal into reality. It’s not mere logic or stuff we build. But it’s not just a bunch of symbols either. It is a set of tools. It’s important not to confuse the model with reality. It’s important not to confuse the tool with reality. We get caught up with the tool when we stop thinking about it as a tool. Since there are lots of tools, there is more than one way to attack a problem. Tools help us think—we shouldn’t waste too much time trying to defend a tool—the idea is simply to use the tool and see where we end up. Tools help us think, but on the other hand they can’t replace thinking. I think the idea of the GBM makes it easier to see that we cannot fit everything we are doing into one category.
So the GBM is bigger than attempts to systematize it. Somehow the GBM stretches beyond whatever the GBM has been up to now—it (if we can call it an *it*) tries new things.

I think Newton is right that mathematics is something we have to cultivate—i.e. the toolkit keeps changing—Socrates is not operating with the same tools that we are—and so we have to keep coming up with new resources and new tools. Experimenting—trying new things—is the recombinant DNA of everything that the GBM has come up with up to now. At so the end of this reasoning is the insight that there are some things that are just beyond what we are able to reason about up to now—i.e, there are some things that the GBM has not found a way to detect.

This leads us to some important conclusions and this is where I will finish up in my reflections. First of all, mathematics is a tool and does not tell us what to use it for. Intelligence is logically distinct from motivation: an ability to figure out how best to get from A to B says nothing about what to do when you get there. Reason can be a means to attaining our most important goals (e.g. seeing the truth) but can also be co-opted into mere means-to-end calculations. Husserl, Heidegger, Arendt all saw this—they all tried to pry apart the cheap version of understanding (which they all call "positivism") from a more sophisticated one (which Heidegger, e.g. calls "meditative thinking").

This result leads to the last reflection. In his *Art of Discovery*, from 1685, after having invented the calculus, Leibniz expressed the hope that one day a truly *universal* form of calculation might replace opinion and thus put an end to all disagreement:

"The only way to rectify our reasonings is to make them as tangible as those of the mathematicians, so that we can find our error at a glance, and when there are disputes among persons, we can simply say: Let us simply calculate, to see who is right."

Great as he was, what Leibniz says here represents a fundamental mistake in thinking about the subject. Disagreements are as fundamental to mathematics as are computing sums. Disagreeing is another kind of mathematical tool—another 'data collection device'—another way of sending out some signals and getting some feedback when they come back. The point is that we cannot replace judgment with mere calculation.

We cannot endow any system with everything we have learned, with all our good sense and experience, and thus be done with human judgment. To try to do this would be

like trying to replace wisdom with a kind of machine—which would be insane. In a way, the world we live in
is a world constantly flirting with
exactly this insanity—we are trying to make things so easy for ourselves that
we risk farming out our very soul. There
is no Copenhagen solution for philosophy. There is no point in just going on,
even if we don't know what we are doing. The point is to know *exactly* what we are doing, and to try to
act for the good. Therefore it is essential that we focus on our own *human*-friendly purposes and not on
replacing ourselves with any *machine* intelligence.

The tools are there for us to use—theyare not there to *spare* us the problem of thinking, but to *free it up*—we just have to get back to the process of seeing
where a hypothesis gets us. And it is also worth saying that *the most important tool we have* is our
moral sensibility—the question,

what is this for?—why are we doing this?—what do we hope
to accomplish?

It's important to stress that we are not locked in to
looking at something happening in the world in one and only one way— but
this is not a problem
but actually an advantage. We only get
to see what we think about, what we ask about and examine— we have lots of ways of doing this—and it seems very clear that there is more to
see than any one method can ever reveal. The point is to go on with our
problem-seeking, constructive, experimental, imaginative way of interacting with the world, rather than
trading in the toolbox—since it's just a box of tools—with the conviction that
there is only one truth—that there
is only one reality—that there is only one thing to see and one way to see it.

So to conclude, there is no general answer to the question,
what is mathematics?, because the GBM keeps evolving—the GBM is what we make it—which means that pretty
much everything we have

been taught about this subject is wrong. We don't have a clue what mathematics is. Minimally it is many things. It is not at all obvious what we should do with it. We don't know which tool to use at which moment. And so the big conclusion from studying the philosophy of mathematics is to go on with life—to go on learning—to go on cultivating mathematics—to go on in everything we are trying to interact with, understand and improve—as a form of philosophical questioning.

Notes

On the background for Pythagoras, see Proclus, *Commentary on Euclid*. On the background for Mathematical Platonism, see:

A History of Greek Mathematics, Thomas Heath (Oxford: Clarendon Press, 1921).

The Development of Logic, William Kneale and Martha Kneale (Oxford: Clarendon Press, 1962), Chapter 3.

Greek Mathematical Thought and the Origin of Algebra, Jacob Klein (Cambridge, Mass.: M.I.T. Press, 1968).

Øystein Linnebo, https://plato.stanford.edu/entries/ platonism-mathematics/, 2009.

Regarding Husserl, see:

Edmund Husserl, *Cartesian Meditations* (The Hague: Martinus Nijhoff, 1931), § 64.

Edmund Husserl, *Crisis of the European Sciences* (The Hague: Martinus Nijhoff, 1954), II, g.

"Mathematics and mathematical science, as a garb of ideas, or the garb of symbols of the symbolic mathematical theories, encompasses everything which, for scientists and the educated generally, represents the life-world, and dresses it up as objectively actual and true nature. It is this garb of ideas that we take for true being. But this is actually only a method. Thus Galileo is a discoverer who also conceals a great deal" (ii, h).

Robert Sokolowski, "Edmund Husserl and the Principles of Phenomenology," in the collection edited by John C. Ryan, *Twentieth Century Thinkers* (Staten Island: Alba House, 1965), pp. 134-157.

Regarding Heidegger:

Martin Heidegger, *What is a Thing?,* translation by Barton and Deutsch (Chicago: Gateway, 1967), Part B, 1, section 5 (pp. 65-107).

"*Ta mathemata*, mathematical things, are things insofar as we take cognizance of them as what we already know them to be in advance. Thus we already know in advance what counts as a physical object, we already know the plant-like character of plants, we already have an idea of what counts as an animal and the animal-like characteristics that alert us that what we are looking at is an animal. Thus the kind of learning that we are talking about—*ta mathemata*, things insofar as they can be learned —- is an extremely peculiar kind of seeing and taking in which we are taking something *we already have* " (73).

"The mathematical is the fundamental presupposition of the knowledge of things" (75). Martin Heidegger, *Beiträge zur Philosophie* (Frankfurt: Klosterman, 1989), 2:6, 4:8, 66:91

Regarding Arendt:

Hannah Arendt, *The Human Condition* (Chicago: University Press, 1958). Arendt quotes Alexander Koyré, Karl Jaspers, Max Weber, Alfred North Whitehead, E.A. Burtt's *Metaphysical Foundations of Modern Science*, Ernst Cassirer, Jacob Bronowski, and thinking from working scientists of her day including Albert Einstein and Werner Heisenberg.

"In the experiment man realized his newly won freedom from the shackles of earth-bound experience; instead

of observing natural phenomena as they were given to him, man placed nature under the conditions of his own mind, that is, under conditions won from a universal, astrophysical viewpoint, a cosmic standpoint outside nature itself" (265).

Further:

Anton Chekhov, *Notebooks of 1921*: "there is no national

multiplication table."
Wittgenstein ends up in the
untenable place of
social constructionist idea mathematics in his

Remarks
on the Foundations of Mathematics (notes
from 1937-44; see e.g. note VI, §67, 72) and in *On Certainty* (notes from 1950-51, e.g. § 204).

Wittgenstein's student R. L. Goodstein
introduced the new operation "tetration" in 1947.

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